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# THE DENSITY OF NUMBERS $n$ HAVING A PRESCRIBED G.C.D. WITH THE $n$ TH FIBONACCI NUMBER

CARLO SANNA AND EMANUELE TRON

ABSTRACT. For each positive integer  $k$ , let  $\mathcal{A}_k$  be the set of all positive integers  $n$  such that  $\gcd(n, F_n) = k$ , where  $F_n$  denotes the  $n$ th Fibonacci number.

We prove that the asymptotic density of  $\mathcal{A}_k$  exists and is equal to

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{\text{lcm}(dk, z(dk))}$$

where  $\mu$  is the Möbius function and  $z(m)$  denotes the least positive integer  $n$  such that  $m$  divides  $F_n$ . We also give an effective criterion to establish when the asymptotic density of  $\mathcal{A}_k$  is zero and we show that this is the case if and only if  $\mathcal{A}_k$  is empty.

## 1. INTRODUCTION

Let  $(u_n)_{n \geq 1}$  be a nondegenerate linear recurrence with integral values. The arithmetic relations between  $u_n$  and  $n$  are a topic which has attracted the attention of several researchers, especially in recent years. For instance, the set of positive integers  $n$  such that  $u_n$  is divisible by  $n$  has been studied by Alba González, Luca, Pomerance, and Shparlinski [1], under the mild hypothesis that the characteristic polynomial of  $(u_n)_{n \geq 1}$  has only simple roots; and by André-Jeannin [2], Luca and Tron [12], Somer [17], and Sanna [16], when  $(u_n)_{n \geq 1}$  is a Lucas sequence. A problem in a sense dual to this is that of understanding when  $n$  is coprime to  $u_n$ . In this respect, Sanna [15, Theorem 1.1] recently proved the following result.

**Theorem 1.1.** *The set of positive integers  $n$  such that  $\gcd(n, u_n) = 1$  has a positive asymptotic density, unless  $(u_n/n)_{n \geq 1}$  is a linear recurrence.*

In this paper, we focus on the linear recurrence of Fibonacci numbers  $(F_n)_{n \geq 1}$ , defined as usual by  $F_1 = F_2 = 1$  and  $F_{n+2} = F_{n+1} + F_n$  for all integers  $n \geq 1$ . For each positive integer  $k$ , define the set

$$\mathcal{A}_k := \{n \geq 1 : \gcd(n, F_n) = k\}.$$

Leonetti and Sanna [11, Theorems 1.1 and 1.3] proved the following:

**Theorem 1.2.** *If  $\mathcal{B} := \{k \geq 1 : \mathcal{A}_k \neq \emptyset\}$  then its counting function satisfies*

$$\#\mathcal{B}(x) \gg \frac{x}{\log x},$$

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for all  $x \geq 2$ . Furthermore,  $\mathcal{B}$  has zero asymptotic density.

Let  $z(m)$  be the *rank of appearance*, or *entry point*, of a positive integer  $m$  in the sequence of Fibonacci numbers, that is, the smallest positive integer  $n$  such that  $m$  divides  $F_n$ . It is well known that  $z(m)$  exists. Set also  $\ell(m) := \text{lcm}(m, z(m))$ .

Our first result establishes the existence of the asymptotic density of  $\mathcal{A}_k$  and provides an effective criterion to check whether this asymptotic density is positive.

**Theorem 1.3.** *For each positive integer  $k$ , the asymptotic density of  $\mathcal{A}_k$  exists. Moreover,  $\mathbf{d}(\mathcal{A}_k) > 0$  if and only if  $\mathcal{A}_k \neq \emptyset$  if and only if  $k = \gcd(\ell(k), F_{\ell(k)})$ .*

Our second result is an explicit formula for the asymptotic density of  $\mathcal{A}_k$ .

**Theorem 1.4.** *For each positive integer  $k$ , we have*

$$\mathbf{d}(\mathcal{A}_k) = \sum_{d=1}^{\infty} \frac{\mu(d)}{\ell(dk)}, \quad (1)$$

where  $\mu$  is the Möbius function.

**Notation.** Throughout, we reserve the letters  $p, q, r$  for prime numbers. For a set of positive integers  $\mathcal{S}$ , we put  $\mathcal{S}(x) := \mathcal{S} \cap [1, x]$  for all  $x \geq 1$ , and we recall that the asymptotic density  $\mathbf{d}(\mathcal{S})$  of  $\mathcal{S}$  is defined as the limit of the ratio  $\#\mathcal{S}(x)/x$ , as  $x \rightarrow +\infty$ , whenever this exists. As usual,  $\mu(n)$ ,  $\varphi(n)$ ,  $\omega(n)$ , and  $\tau(n)$ , denote the Möbius function, the Euler's totient function, the number of distinct prime factors, and the number of divisors of a positive integer  $n$ , respectively. We employ the Landau–Bachmann “Big Oh” and “little oh” notations  $O$  and  $o$ , as well as the associated Vinogradov symbol  $\ll$ .

## 2. PRELIMINARIES

The next lemma summarizes some basic properties of  $\ell$ ,  $z$ , and the Fibonacci numbers, which we will implicitly use later without further mention.

**Lemma 2.1.** *For all positive integers  $m, n$  and all prime numbers  $p$ , we have:*

- (i)  $m \mid F_n$  if and only if  $z(m) \mid n$ .
- (ii)  $z(\text{lcm}(m, n)) = z(\ell(m), \ell(n))$ .
- (iii)  $z(p) \mid p - \left(\frac{p}{5}\right)$ , where  $\left(\frac{p}{5}\right)$  is a Legendre symbol.
- (iv)  $\nu_p(F_n) \geq \nu_p(n)$  whenever  $z(p) \mid n$ .
- (v)  $m \mid \gcd(n, F_n)$  if and only if  $\ell(m) \mid n$ .
- (vi)  $\ell(\text{lcm}(m, n)) = \text{lcm}(\ell(m), \ell(n))$ .
- (vii)  $\ell(p) = pz(p)$  for  $p \neq 5$ , while  $\ell(5) = 5$ .

*Proof.* Facts (i)–(iii) are well-known (see, e.g., [13]). Fact (iv) follows quickly from the formulas for  $\nu_p(F_n)$  given by Lengyel [10]. Finally, (v)–(vii) are easy consequences of (i)–(iii) and the definition of  $\ell$ .  $\square$

Now we state an easy criterion to establish if  $\mathcal{A}_k \neq \emptyset$  [11, Lemma 2.2(iii)].

**Lemma 2.2.**  *$\mathcal{A}_k \neq \emptyset$  if and only if  $k = \gcd(\ell(k), F_{\ell(k)})$ , for all integers  $k \geq 1$ .*

If  $\mathcal{S}$  is a set of positive integers, we define its *set of nonmultiples* as

$$\mathcal{N}(\mathcal{S}) := \{n \geq 1 : s \nmid n \text{ for all } s \in \mathcal{S}\}.$$

Sets of nonmultiples, or more precisely their complement *sets of multiples*

$$\mathcal{M}(\mathcal{S}) := \{n \geq 1 : s \mid n \text{ for some } s \in \mathcal{S}\},$$

have been studied by several authors, we refer the reader to [7] for a systematic treatment of this topic. We shall need only the following result.

**Lemma 2.3.** *If  $\mathcal{S}$  is a set of positive integers such that*

$$\sum_{s \in \mathcal{S}} \frac{1}{s} < +\infty,$$

*then  $\mathcal{N}(\mathcal{S})$  has an asymptotic density. Moreover, if  $1 \notin \mathcal{S}$  then  $\mathbf{d}(\mathcal{N}(\mathcal{S})) > 0$ .*

*Proof.* The part about the existence of  $\mathbf{d}(\mathcal{N}(\mathcal{S}))$  is due to Erdős [5], while the second assertion follows easily from the inequality

$$\mathbf{d}(\mathcal{N}(\mathcal{S})) \geq \prod_{s \in \mathcal{S}} \left(1 - \frac{1}{s}\right)$$

proved by Heilbronn [8] and Rohrbach [14]. □

For any  $\gamma > 0$ , let us define

$$\mathcal{Q}_\gamma := \{p : z(p) \leq p^\gamma\}.$$

The following is a well-known lemma, which belongs to the folklore.

**Lemma 2.4.** *For all  $x, \gamma > 0$ , we have  $\#\mathcal{Q}_\gamma(x) \ll x^{2\gamma}$ .*

*Proof.* It is enough noting that

$$2^{\#\mathcal{Q}_\gamma(x)} \leq \prod_{p \in \mathcal{Q}_\gamma(x)} p \mid \prod_{n \leq x^\gamma} F_n \leq 2^{\sum_{n \leq x^\gamma} n} = 2^{O(x^{2\gamma})},$$

where we employed the inequality  $F_n \leq 2^n$ , valid for all positive integers  $n$ . □

As usual, for  $x \geq y \geq 0$  let  $\Psi(x, y)$  be the number of positive integers  $n \leq x$  which are *y-smooth*, or *y-friable*, that is, they have no prime factor greater than  $y$ . We need the following estimate for  $\Psi(x, y)$ , which is a straightforward consequence of [3, Corollary to Theorem 3.1].

**Lemma 2.5.** *For  $x \geq y > 1$ , we have*

$$\Psi(x, y) = x \exp(-(1 + o(1)) u \log u)$$

*uniformly in the range  $y \geq (\log x)^2$  as long as  $u \rightarrow +\infty$ , where*

$$u := \frac{\log x}{\log y}.$$

The next lemma is a bound regarding prime numbers in arithmetic progressions [9, Lemma 6].

**Lemma 2.6.** *We have*

$$\sum_{\substack{p \equiv a \pmod{b} \\ b \leq p \leq x}} \frac{1}{p} \ll \frac{\log \log x}{\varphi(b)}$$

uniformly for all  $x \geq 3$  and all relatively prime integers  $a$  and  $b$  with  $0 < b \leq x$ .

We conclude with a lemma about positive integers with many prime factors.

**Lemma 2.7.** *For  $x, t \geq 2$ , the number of positive integers  $n$  such that  $\omega(n) \geq t$  is*

$$O\left(\frac{x \log x}{2^t}\right).$$

*Proof.* Call  $\mathcal{S}$  the set of positive integers  $n$  such that  $\omega(n) \geq t$ . Then,

$$2^t \cdot \#\mathcal{S}(x) \leq \sum_{n \in \mathcal{S}(x)} 2^{\omega(n)} \leq \sum_{n \in \mathcal{S}(x)} \tau(n) \leq \sum_{n \leq x} \tau(n) \ll x \log x,$$

where the last bound is well known [4, Theorem 4.9]. The claim follows.  $\square$

### 3. PROOF OF THEOREM 1.3

We begin by showing that  $\mathcal{A}_k$  is a scaled set of nonmultiples.

**Lemma 3.1.** *For each positive integer  $k$  such that  $\mathcal{A}_k \neq \emptyset$ , we have*

$$\mathcal{A}_k = \{\ell(k)m : m \in \mathcal{N}(\mathcal{L}_k)\},$$

where

$$\mathcal{L}_k := \{p : p \mid k\} \cup \{\ell(kp)/\ell(k) : p \nmid k\}.$$

*Proof.* We know that  $n \in \mathcal{A}_k$  implies  $\ell(k) \mid n$ , hence it is enough to prove that  $\ell(k)m \in \mathcal{A}_k$ , for some positive integer  $m$ , if and only if  $m \in \mathcal{N}(\mathcal{L}_k)$ .

Clearly,  $\ell(k)m \in \mathcal{A}_k$  for some positive integer  $m$ , if and only if

$$\nu_p(\gcd(\ell(k)m, F_{\ell(k)m})) = \nu_p(k) \tag{2}$$

for all prime numbers  $p$ .

Let  $p$  be a prime number dividing  $k$ . Then, for all positive integer  $m$ , we have  $z(p) \mid \ell(k)m$  and consequently  $\nu_p(F_{\ell(k)m}) \geq \nu_p(\ell(k)m)$ , so that

$$\nu_p(\gcd(\ell(k)m, F_{\ell(k)m})) = \nu_p(\ell(k)m) = \nu_p(\ell(k)) + \nu_p(m). \tag{3}$$

In particular, recalling that  $k = \gcd(\ell(k), F_{\ell(k)})$  since  $\mathcal{A}_k \neq \emptyset$  and thanks to Lemma 2.2, for  $m = 1$  we get

$$\nu_p(k) = \nu_p(\gcd(\ell(k), F_{\ell(k)})) = \nu_p(\ell(k)),$$

which together with (3) gives

$$\nu_p(\gcd(\ell(k)m, F_{\ell(k)m})) = \nu_p(k) + \nu_p(m). \tag{4}$$

Therefore, (2) holds if and only if  $p \nmid m$ .

Now let  $p$  be a prime number not dividing  $k$ . Then (2) holds if and only if

$$p \nmid \gcd(\ell(k)m, F_{\ell(k)m}),$$

that is,  $\ell(p) \nmid \ell(k)m$ , which in turn is equivalent to

$$\frac{\ell(kp)}{\ell(k)} = \frac{\text{lcm}(\ell(k), \ell(p))}{\ell(k)} \nmid m,$$

since  $p$  and  $k$  are relatively prime.

Summarizing, we have found that  $\ell(k)m \in \mathcal{A}_k$ , for some positive integer  $m$ , if and only if  $p \nmid m$  for all prime numbers  $p$  dividing  $k$ , and  $\ell(kq)/\ell(k) \nmid m$  for all prime numbers  $q$  not dividing  $k$ , that is,  $m \in \mathcal{N}(\mathcal{L}_k)$ .  $\square$

Now if  $k$  is a positive integer such that  $\mathcal{A}_k = \emptyset$  then, obviously, the asymptotic density of  $\mathcal{A}_k$  exists and is equal to zero. So we can assume  $\mathcal{A}_k \neq \emptyset$ , which in turn, by Lemma 2.2, implies that  $k = \gcd(\ell(k), F_{\ell(k)})$ .

Henceforth, the implied constants may depend on  $k$ . We have

$$\sum_{n \in \mathcal{L}_k} \frac{1}{n} \ll \sum_p \frac{1}{\ell(kp)} \leq \sum_p \frac{1}{\ell(p)} \ll \sum_p \frac{1}{pz(p)}. \quad (5)$$

Fix any  $\gamma \in ]0, 1/2[$ . On the one hand,

$$\sum_{p \notin \mathcal{Q}_\gamma} \frac{1}{pz(p)} < \sum_{p \notin \mathcal{Q}_\gamma} \frac{1}{p^{1+\gamma}} < \sum_n \frac{1}{n^{1+\gamma}} < +\infty. \quad (6)$$

On the other hand, by partial summation and by Lemma 2.4,

$$\begin{aligned} \sum_{p \in \mathcal{Q}_\gamma} \frac{1}{pz(p)} &< \sum_{p \in \mathcal{Q}_\gamma} \frac{1}{p} = \frac{\#\mathcal{Q}_\gamma(t)}{t} \Big|_{t=2}^{+\infty} + \int_2^{+\infty} \frac{\#\mathcal{Q}_\gamma(t)}{t^2} dt \\ &\ll \int_2^{+\infty} \frac{dt}{t^{2-2\gamma}} < +\infty. \end{aligned} \quad (7)$$

Therefore, putting together (5), (6), and (7), we get

$$\sum_{n \in \mathcal{L}_k} \frac{1}{n} < +\infty,$$

while clearly  $1 \notin \mathcal{L}_k$ . Hence, Lemma 2.3 tell us that  $\mathcal{N}(\mathcal{L}_k)$  has a positive asymptotic density. Finally, by Lemma 3.1 we conclude that the asymptotic density of  $\mathcal{A}_k$  exists and it is positive.

*Remark 3.2.* We note that the convergence of the series

$$\sum_p \frac{1}{\ell(p)} \quad (8)$$

also follows from the stronger, and independent, result of Corollary 4.3 in the next section. However, since the proof of Corollary 4.3 is quite involved, we preferred to give an easier proof of the convergence of (8) in this section.

## 4. PROOF OF THEOREM 1.4

For the sake of convenience, put

$$L(n) := \prod_{p|n} \ell(p)$$

for all positive integers  $n$ . We start with a lower bound for  $L(n)$ .

**Lemma 4.1.** *Pick  $\delta > 0$ . Then there exists  $\varepsilon = \varepsilon(\delta) > 0$  such that, for all  $x > 1$ , the inequality  $L(n) > n^{1+\varepsilon}$  holds for all squarefree positive integers  $n \leq x$  but*

$$O\left(\frac{x}{(\log x)^\delta}\right)$$

*exceptions.*

*Proof.* Set  $\alpha := \max\{2, 3\delta\}$  and  $y := (\log x)^\alpha$ . Then, by Lemma 2.5, we have

$$\Psi(x, y) = x^{1-1/\alpha+o(1)},$$

as  $x \rightarrow +\infty$ . Put  $\gamma = 1/3$  and let  $\mathcal{S}_1$  be the set of positive integers  $n$  such that there exists  $p \in \mathcal{Q}_\gamma$  with  $p > y$  and  $p | n$ . We have

$$\begin{aligned} \#\mathcal{S}_1(x) &\leq \sum_{\substack{p \in \mathcal{Q}_\gamma \\ p > y}} \frac{x}{p} = x \left[ \frac{\#\mathcal{Q}_\gamma(t)}{t} \Big|_{t=y}^{+\infty} + \int_y^{+\infty} \frac{\#\mathcal{Q}_\gamma(t)}{t^2} dt \right] \\ &\ll x \int_y^{+\infty} \frac{dt}{t^{2-2\gamma}} \ll \frac{x}{y^{1/3}} \leq \frac{x}{(\log x)^\delta}. \end{aligned}$$

For the rest of the proof, write  $n = mM$  where  $m$  is the  $y$ -smooth part of  $n$  and, consequently, each prime factor of  $M$  is greater than  $y$ . Let  $\mathcal{S}_2$  be the set of positive integers  $n$  such that  $M \leq n^{1/(2\alpha)}$ . Suppose  $n \in \mathcal{S}_2(x)$ . On the one hand, since  $m$  is  $y$ -smooth, there are at most  $\Psi(x, y) = x^{1-1/\alpha+o(1)}$  choices for  $m$ . On the other hand,  $M \leq n^{1/(2\alpha)} \leq x^{1/(2\alpha)}$ , so obviously there are at most  $x^{1/(2\alpha)}$  choices for  $M$ . It follows that

$$\#\mathcal{S}_2(x) \leq x^{1-1/\alpha+o(1)} \cdot x^{1/(2\alpha)} = x^{1-1/(2\alpha)+o(1)} \ll \frac{x}{(\log x)^\delta}.$$

Now, if  $n$  is squarefree and does not belong to  $\mathcal{S}_1(x)$  nor  $\mathcal{S}_2(x)$ , then

$$L(n) \geq n \prod_{\substack{p|n \\ p > y}} z(p) > n \prod_{\substack{p|n \\ p > y}} p^\gamma > nM^\gamma > n^{1+\gamma/(2\alpha)} = n^{1+\varepsilon},$$

where  $\varepsilon := \gamma/(2\alpha) > 0$ . □

Now we give a lower bound for  $\ell(n)$ . The proof proceeds similarly to the one of [6, Theorem 5] but with a few adjustments.

**Lemma 4.2.** *Pick  $\delta \geq 1$ . Then there exists  $\varepsilon = \varepsilon(\delta) > 0$  such that, for all  $x > 1$ , the inequality  $\ell(n) > n^{1+\varepsilon}$  holds for all squarefree positive integers  $n \leq x$  but*

$$O\left(\frac{x}{(\log x)^\delta}\right)$$

exceptions.

*Proof.* Clearly, we can suppose  $x$  sufficiently large.

Pick  $\varepsilon' > 0$  sufficiently small, depending on  $\delta$ , and let  $\mathcal{S}_1$  be the set of square-free positive integers  $n$  such that  $L(n) \leq n^{1+\varepsilon'}$ . Then, by Lemma 4.1, we have

$$\#\mathcal{S}_1(x) \ll \frac{x}{(\log x)^\delta}.$$

Put  $y := (\log x)^{\delta+1}$  and let  $\mathcal{S}_2$  be the set of squarefree positive integers  $n$  such that  $p^2 \mid L(n)$  for some prime number  $p > y$ . If  $n \in \mathcal{S}_2(x)$ , then one of the following three cases occurs:

- (i)  $p^2 \mid z(q)$  and  $q \mid n$ ;
- (ii)  $p \mid z(q)$  and  $pq \mid n$ ;
- (iii)  $p \mid z(q)$ ,  $p \mid z(r)$ , and  $qr \mid n$ ;

where  $p, q, r$  are prime numbers with  $p > y$ . The number of positive integers  $n \leq x$  such that case (iii) occurs is at most

$$\begin{aligned} \sum_{p>y} \sum_{\substack{q,r \equiv \pm 1 \pmod p \\ q,r \leq x}} \frac{x}{qr} &\leq x \sum_{p>y} \left( \sum_{\substack{q \equiv \pm 1 \pmod p \\ q \leq x}} \frac{1}{q} \right)^2 \ll x (\log \log x)^2 \sum_{p>y} \frac{1}{p^2} \\ &\ll \frac{x (\log \log x)^2}{y} \ll \frac{x}{(\log x)^\delta}, \end{aligned}$$

where we used Lemma 2.6. Cases (i) and (ii) can be treated in essentially the same way, so we obtain

$$\#\mathcal{S}_2(x) \ll \frac{x}{(\log x)^\delta}.$$

Put  $C := (\varepsilon' \log 2)/8$  and  $w := x^{C/\log y}$ . Let  $\mathcal{R}$  be the set of prime numbers  $p$  such that the  $y$ -smooth part of  $p - (\frac{p}{5})$  is greater than  $w$ . Also, let  $\mathcal{F}$  be the set of  $y$ -smooth positive integers in  $[w, x]$ . By Lemma 2.5 and a little computation, we have

$$\Psi(t, y) \leq t^{1-c}$$

for all  $t \in [w, x]$ , where  $c > 0$  is a constant. Therefore,

$$\begin{aligned} \sum_{k \in \mathcal{F}} \frac{1}{k} &= \frac{\Psi(t, y)}{t} \Big|_{t=w}^x + \int_w^x \frac{\Psi(t, y)}{t^2} dt \leq \Psi(x, y) + \int_w^x t^{-c-1} dt \\ &\ll \frac{1}{x^c} + \frac{1}{w^c} \ll \frac{1}{(\log x)^{\delta+1}}. \end{aligned} \tag{9}$$

Now let  $\mathcal{S}_3$  be the set of positive integers  $n$  which have a prime factor in  $\mathcal{R}$ . By Lemma 2.6 and (9), we get

$$\begin{aligned} \#\mathcal{S}_3(x) &\leq \sum_{p \in \mathcal{R}} \frac{x}{p} = x \sum_{k \in \mathcal{F}} \sum_{\substack{p \equiv \pm 1 \pmod k \\ p \leq x}} \frac{1}{p} \ll x \log \log x \sum_{k \in \mathcal{F}} \frac{1}{\varphi(k)} \\ &\ll x (\log \log x)^2 \sum_{k \in \mathcal{F}} \frac{1}{k} \ll \frac{x}{(\log x)^\delta}, \end{aligned}$$



where we also employed the inequality  $\varphi(k) \gg k/\log \log k$  [4, Proposition 8.4].

Let  $\mathcal{S}_4$  be the set of positive integers  $n$  such that  $\omega(n) \geq \log y/\log 2$ . Then Lemma 2.7 yields

$$\#\mathcal{S}_4(x) \ll \frac{x \log x}{y} = \frac{x}{(\log x)^\delta}.$$

Finally, pick a squarefree positive integer

$$n \notin [1, x^{1/2}] \cup \bigcup_{i=1}^4 \mathcal{S}_i$$

and write  $L(n) = mM$ , where  $m$  is  $y$ -smooth and all prime factors of  $M$  are greater than  $y$ . Since  $n \notin \mathcal{S}_2$ , we have that  $M$  is squarefree, which in turn implies that  $M \mid \ell(n)$ . Furthermore, by  $n \notin \mathcal{S}_3 \cup \mathcal{S}_4$  and since  $n$  is squarefree,

$$\begin{aligned} m &\leq \prod_{\substack{q \mid n \\ q \leq y}} q \cdot \prod_{p \mid n} \left( y\text{-smooth part of } p - \left(\frac{p}{y}\right) \right) \leq y^{\omega(n)} \cdot w^{\omega(n)} \\ &< w^{2\omega(n)} < x^{2C/\log 2} < n^{4C/\log 2} = n^{\varepsilon'/2}, \end{aligned}$$

where we also used the fact that  $y < w$  for sufficiently large  $x$ . Therefore,

$$\ell(n) \geq M = \frac{L(n)}{m} > \frac{n^{1+\varepsilon'}}{m} > n^{1+\varepsilon}$$

since  $n \notin \mathcal{S}_1$ , where  $\varepsilon := \varepsilon'/2 > 0$ , and the proof is complete.  $\square$

From Lemma 4.2 we obtain the following corollary.

**Corollary 4.3.** *We have*

$$\sum_{d=1}^{\infty} \frac{|\mu(d)|}{\ell(d)} < +\infty.$$

*Proof.* Fix some  $\delta > 1$  and let  $\varepsilon$  be given by Lemma 4.2. Also, let  $\mathcal{E}$  be the set of squarefree positive integers  $d$  such that  $\ell(d) \leq d^{1+\varepsilon}$ . On the one hand,

$$\sum_{d \notin \mathcal{E}} \frac{|\mu(d)|}{\ell(d)} < \sum_{d=1}^{\infty} \frac{1}{d^{1+\varepsilon}} < +\infty.$$

On the other hand, by partial summation and by Lemma 4.2,

$$\begin{aligned} \sum_{d \in \mathcal{E}} \frac{|\mu(d)|}{\ell(d)} &< \sum_{d \in \mathcal{E}} \frac{1}{d} = \frac{\#\mathcal{E}(t)}{t} \Big|_{t=2}^{+\infty} + \int_2^{+\infty} \frac{\#\mathcal{E}(t)}{t^2} dt \\ &\ll \int_2^{+\infty} \frac{dt}{t(\log t)^\delta} < +\infty. \end{aligned}$$

The claim follows.  $\square$

Now we shall introduce a family of sets. For each positive integer  $k$ , let  $\mathcal{B}_k$  be the set of positive integers  $n$  such that:

- (i)  $k \mid \gcd(n, F_n)$ ;
- (ii) if  $p \mid \gcd(n, F_n)$  for some prime number  $p$ , then  $p \mid k$ .

The essential part of the proof of Theorem 1.4 is the following formula for the asymptotic density of  $\mathcal{B}_k$ .

**Lemma 4.4.** *For all positive integers  $k$ , the asymptotic density of  $\mathcal{B}_k$  exists and*

$$\mathbf{d}(\mathcal{B}_k) = \sum_{(d,k)=1} \frac{\mu(d)}{\ell(dk)}, \quad (10)$$

where the series is absolutely convergent.

*Proof.* For all positive integers  $n$  and  $d$ , let us define

$$\varrho(n, d) := \begin{cases} 1 & \text{if } d \mid F_n, \\ 0 & \text{if } d \nmid F_n. \end{cases}$$

Note that  $\varrho$  is multiplicative in its second argument, that is,

$$\varrho(n, de) = \varrho(n, d)\varrho(n, e)$$

for all relatively prime positive integers  $d$  and  $e$ , and all positive integers  $n$ .

It is easy to see that  $n \in \mathcal{B}_k$  if and only if  $\ell(k) \mid n$  and  $\varrho(n, p) = 0$  for all prime numbers  $p$  dividing  $n$  but not dividing  $k$ . Therefore,

$$\begin{aligned} \#\mathcal{B}_k(x) &= \sum_{\substack{n \leq x \\ \ell(k) \mid n}} \prod_{\substack{p \mid n \\ p \nmid k}} (1 - \varrho(n, p)) = \sum_{\substack{n \leq x \\ \ell(k) \mid n}} \sum_{\substack{d \mid n \\ (d,k)=1}} \mu(d) \varrho(n, d) \\ &= \sum_{\substack{d \leq x \\ (d,k)=1}} \mu(d) \sum_{\substack{m \leq x/d \\ \ell(k) \mid dm}} \varrho(dm, d), \end{aligned} \quad (11)$$

for all  $x > 0$ . Moreover, given a positive integer  $d$  which is relatively prime with  $k$ , we have that  $\varrho(dm, d) = 1$  and  $\ell(k) \mid dm$  if and only if  $\text{lcm}(z(d), \ell(k)) \mid dm$ , which in turn is equivalent to  $m$  being divisible by

$$\frac{\text{lcm}(d, \text{lcm}(z(d), \ell(k)))}{d} = \frac{\text{lcm}(\ell(d), \ell(k))}{d} = \frac{\ell(dk)}{d},$$

since  $d$  and  $k$  are relatively prime. Hence,

$$\sum_{\substack{m \leq x/d \\ \ell(k) \mid dm}} \varrho(dm, d) = \sum_{\substack{m \leq x/d \\ \ell(dk)/d \mid m}} 1 = \left\lfloor \frac{x}{\ell(dk)} \right\rfloor,$$

for all  $x > 0$ , which together with (11) implies that

$$\#\mathcal{B}_k(x) = \sum_{\substack{d \leq x \\ (d,k)=1}} \mu(d) \left\lfloor \frac{x}{\ell(dk)} \right\rfloor = x \sum_{\substack{d \leq x \\ (d,k)=1}} \frac{\mu(d)}{\ell(dk)} - R(x), \quad (12)$$

for all  $x > 0$ , where

$$R(x) := \sum_{\substack{d \leq x \\ (d,k)=1}} \mu(d) \left\{ \frac{x}{\ell(dk)} \right\}.$$

Now, thanks to Corollary 4.3, we have

$$\sum_{(d,k)=1} \frac{|\mu(d)|}{\ell(dk)} \leq \sum_{d=1}^{\infty} \frac{|\mu(d)|}{\ell(d)} < +\infty,$$

hence the series in (10) is absolutely convergent.

It remains only to prove that  $R(x) = o(x)$  as  $x \rightarrow +\infty$ , and then the desired result follows from (12). We have

$$\begin{aligned} |R(x)| &\leq \sum_{d \leq x} |\mu(d)| \left\{ \frac{x}{\ell(dk)} \right\} = O(x^{1/2}) + \sum_{x^{1/2} \leq d \leq x} |\mu(d)| \left\{ \frac{x}{\ell(dk)} \right\} \\ &\leq O(x^{1/2}) + x \sum_{d \geq x^{1/2}} \frac{|\mu(d)|}{\ell(d)} = o(x), \end{aligned}$$

as  $x \rightarrow +\infty$ , since  $\ell(dk) \geq \ell(d)$  and, by Corollary 4.3, the last series is the tail of a convergent series and hence converges to 0 as  $x \rightarrow +\infty$ . The proof is complete.  $\square$

At this point, by the definition of  $\mathcal{B}_k$  and by the inclusion-exclusion principle, it follows easily that

$$\#\mathcal{A}_k(x) = \sum_{d|k} \mu(d) \#\mathcal{B}_{dk}(x),$$

for all  $x > 0$ . Hence, by Lemma 4.4, we get

$$\begin{aligned} \mathbf{d}(\mathcal{A}_k) &= \sum_{d|k} \mu(d) \mathbf{d}(\mathcal{B}_{dk}) = \sum_{d|k} \mu(d) \sum_{(e,dk)=1} \frac{\mu(e)}{\ell(dek)} \\ &= \sum_{d|k} \sum_{(e,k)=1} \frac{\mu(de)}{\ell(dek)} = \sum_{f=1}^{\infty} \frac{\mu(f)}{\ell(fk)}, \end{aligned} \tag{13}$$

since every squarefree positive integer  $f$  can be written in a unique way as  $f = de$ , where  $d$  and  $e$  are squarefree positive integers such that  $d|k$  and  $\gcd(e, k) = 1$ . Also note that the rearrangement of the series in (13) is justified by absolute convergence. The proof of Theorem 1.4 is complete.

*Remark 4.5.* As a consequence of Theorem 1.4, note that if  $\mathcal{A}_k = \emptyset$  (or equivalently if  $k = \gcd(\ell(k), F_{\ell(k)})$ , by Lemma 2.2) then the series in (1) evaluates to 0, which is not obvious a priori.

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